A Simple Proof of the “Magic Diagram”

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1 Introduction

In this note, we provide a simple proof of the following “magic diagram.”

Lemma. *The following diagram is a pullback square.*

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow[\Delta_{Y/Z}]{} & Y \times_Z Y \end{array}$$

This diagram is so called the “magic diagram.” I’m not too sure what’s so magical about this diagram, but it is indeed quite useful in algebraic geometry.

Despite the usefulness and versatility of the magic diagram, I couldn’t find a simple proof of it on the Internet. For example, most proofs in Stack Exchange posts (*e.g.*, [1], [2]) explicitly construct the maps and manually check the universal property of the pullback square.

The goal of this note is to provide a simple, almost mechanical proof of the magic diagram purely based on fiber product algebra. The proof is quite simple that I’m pretty sure it must be well-known or trivial to experts, but I couldn’t find it anywhere, so here it is.

2 Ingredients

We will only use the following basic properties of fiber products.

- $X \times_Y Y \cong X$ (cancellation).
- $X \times_A Y \cong Y \times_A X$ (commutativity).
- $(X \times_A Y) \times_B Z \cong X \times_A (Y \times_B Z)$ (associativity).

When applying these rules, one needs to be a bit careful about the morphisms involved in the fiber products, because there may be multiple ways to define fiber products of the same notation. Especially, the associativity rule must be based on the following four morphisms,

$$\begin{array}{ccccccc} X & & & Y & & & Z \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & A & & B & & \end{array}$$

and must not involve any other morphisms (*e.g.*, $X \to B$ or $Y \to A$, if they exist).

That said, in practice, often there is only one way to naturally define the fiber products, so we may not need to worry about this too much.

3 A proof of the magic diagram

Here's a proof of the magic diagram. The main idea is to insert cancellation pairs and do some rearrangement using the commutativity and associativity rules.

Proof.

$$\begin{aligned}
Y \times_{Y \times_Z Y} (X_1 \times_Z X_2) &\cong Y \times_{Y \times_Z Y} ((X_1 \times_Y Y) \times_Z (Y \times_Y X_2)) \\
&\cong Y \times_{Y \times_Z Y} (((X_1 \times_Y Y) \times_Z Y) \times_Y X_2) \\
&\cong Y \times_{Y \times_Z Y} ((X_1 \times_Y (Y \times_Z Y)) \times_Y X_2) \\
&\cong Y \times_{Y \times_Z Y} (((Y \times_Z Y) \times_Y X_1) \times_Y X_2) \\
&\cong Y \times_{Y \times_Z Y} ((Y \times_Z Y) \times_Y (X_1 \times_Y X_2)) \\
&\cong (Y \times_{Y \times_Z Y} (Y \times_Z Y)) \times_Y (X_1 \times_Y X_2) \\
&\cong Y \times_Y (X_1 \times_Y X_2) \\
&\cong X_1 \times_Y X_2,
\end{aligned}$$

While one needs to do some check about the maps used in the associativity rule, I omit the details here since they are just the natural ones. ☐

4 More examples

The trick of inserting cancellation pairs is useful in proving many other non-trivial pullback squares. As an example, here's another useful pullback square in algebraic geometry:

$$\begin{array}{ccc}
X & \xrightarrow{(\mathrm{id}, f)} & X \times_S Y \\
{\scriptstyle f}\downarrow & \lrcorner & \downarrow \\
Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y
\end{array}$$

This diagram is useful when proving the following fact: given a morphism $f : X \to Y$ of S-schemes and a separate S-scheme Y, the graph morphism $(\mathrm{id}, f) : X \to X \times_S Y$ is a closed immersion.

This pullback square can be proven similarly:

Proof.

$$\begin{aligned}
Y \times_{Y \times_S Y} (X \times_S Y) &\cong Y \times_{Y \times_S Y} (Y \times_S X) \\
&\cong Y \times_{Y \times_S Y} (Y \times_S (Y \times_Y X)) \\
&\cong Y \times_{Y \times_S Y} ((Y \times_S Y) \times_Y X) \\
&\cong (Y \times_{Y \times_S Y} (Y \times_S Y)) \times_Y X \\
&\cong Y \times_Y X \\
&\cong X.
\end{aligned}$$

☐

Here's yet another example of a pullback square.

$$\begin{array}{ccc}
X \times_Y W & \longrightarrow & (X \times_Y W) \times_W (X \times_Y W) \\
\downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & X \times_Y X
\end{array}$$

I'll omit the proof of this one, but the proof is quite straightforward with the same trick. Try it yourself!